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**Continuous Stochastic Cellular Automata That Have
a Stationary Distribution and No Detailed Balance**

Tomaso Poggio and Federico Girosi

Abstract

Marroquin and Ramirez (1990) have recently discovered a class of discrete stochastic cellular automata with Gibbsian invariant measure that have a non-reversible dynamic behavior. Practical applications include more powerful algorithms than the Metropolis algorithm to compute MRF models. In this paper we describe a large class of stochastic dynamical systems that has a Gibbs asymptotic distribution but does not satisfy reversibility. We characterize sufficient properties of a sub-class of stochastic differential equations in terms of the associated Fokker-Planck equation for the existence of an asymptotic probability distribution in the system of coordinates which is given. Practical implications include VLSI analog circuits to compute coupled MRF models.

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It is well known (see Stratonovitch, 1963, for instance) that one can associate, under some conditions, to a stochastic continuous automata (i.e., a stochastic differential equation) a so-called Fokker-Planck (F-P) equation in the probability distribution of the state variables. In this note, we wish to characterize conditions under which the F-P equation admits a stationary solution of the Gibbs type.

Let \mathbf{x} a n -dimensional vector of state variables, and $W(\mathbf{x}, t)$ the probability distribution of the state variables described by \mathbf{x} at time t . The F-P equation is:

$$\frac{\partial}{\partial t} W(\mathbf{x}, t) = \left(- \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} d_{\alpha}(\mathbf{x}) + \frac{1}{2} \sum_{\alpha, \beta=1}^n \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} K_{\alpha\beta}(\mathbf{x}) \right) W(\mathbf{x}, t)$$

where $d_{\alpha}(\mathbf{x})$ is the *drift vector* and $K_{\alpha\beta}(\mathbf{x})$ is the *diffusion matrix* (see Stratonovitch, 1963, p. 76).

The stationary solution $w(\mathbf{x})$ of the F-P satisfies the equation:

$$\sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} G_{\alpha}(\mathbf{x}) = 0, \quad (1)$$

where we have defined the *probability current* $G_{\alpha}(\mathbf{x})$:

$$G_{\alpha}(\mathbf{x}) = d_{\alpha}(\mathbf{x})w(\mathbf{x}) - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} K_{\alpha\beta}(\mathbf{x})w(\mathbf{x}). \quad (2)$$

In order to find the stationary solution, we do *not* assume, as Stratonovitch and everybody else does, that $G_{\alpha}(\mathbf{x}) = 0$ and set $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ in equation (1), obtaining:

$$\begin{aligned} & \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} \left(d_{\alpha}(\mathbf{x})e^{-U(\mathbf{x})} - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} e^{-U(\mathbf{x})} K_{\alpha\beta}(\mathbf{x}) \right) = \\ & = \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} e^{-U(\mathbf{x})} \left(d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta}(\mathbf{x}) \frac{\partial}{\partial x_{\beta}} U(\mathbf{x}) - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} K_{\alpha\beta}(\mathbf{x}) \right) = 0 \end{aligned}$$

Assuming that the diffusion matrix is constant, that is $K_{\alpha\beta}(\mathbf{x}) = K_{\alpha\beta}$, we obtain:

$$e^{-U(\mathbf{x})} \sum_{\alpha=1}^n \left[-\frac{\partial}{\partial x_{\alpha}} U(\mathbf{x}) \left(d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta} \frac{\partial}{\partial x_{\beta}} U(\mathbf{x}) \right) + \frac{\partial}{\partial x_{\alpha}} d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta} \frac{\partial^2}{\partial x_{\beta} \partial x_{\alpha}} U(\mathbf{x}) \right] = 0$$

Provided that the diffusion matrix K is invertible we can effectuate the coordinate transformation

$$x_{\alpha} \rightarrow \frac{1}{2} K_{\beta\alpha} x_{\beta}$$

and defining the vector $(\mathbf{d})_{\alpha} = d_{\alpha}(\mathbf{x})$ we rewrite the previous equation as

$$\begin{aligned} -\nabla U \cdot \mathbf{d} - \nabla U \cdot \nabla U + \nabla \cdot \mathbf{d} + \nabla^2 U &= \\ = -\nabla U(\mathbf{d} + \nabla U) + \nabla(\mathbf{d} + \nabla U) &= 0. \end{aligned}$$

We finally obtain

$$(\nabla - \nabla U) \cdot (\nabla U + \mathbf{d}) = 0, \quad (3)$$

which is the condition for stationary distribution.

Thus, one solution is:

$$\nabla U + \mathbf{d} = 0 \Leftrightarrow \mathbf{d} = -\nabla U, \quad (4)$$

which is equivalent to the so called *potential conditions*, that amount to say that \mathbf{d} is the gradient of a potential. If the potential conditions are satisfied the probability current $G_{\alpha}(\mathbf{x})$ is identically zero, and thus *detailed balance* holds. Therefore we recover the well known result that detailed balance implies the existence of a stationary Gibbs distribution $w(\mathbf{x}) = e^{-U(\mathbf{x})}$. However condition (3) shows that the converse is not true. In fact equation (3) has also the solution

$$(\nabla - \nabla U) \cdot \mathbf{f} = 0, \quad (5)$$

with $\mathbf{f} = \nabla U + \mathbf{d}$ and this solution is *not trivial only if* $\mathbf{f} \neq 0$, that is if \mathbf{d} is not the gradient of a function. Equation (3) has therefore a "larger" space of solutions than the one represented by the potential conditions. Of course in

both cases the solution U must be such that $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ is a probability distribution, and therefore the following additional condition must hold:

$$\int d\mathbf{x} e^{-U(\mathbf{x})} < \infty$$

A simple and interesting example that proves the existence of non-trivial solutions U such that $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ is the following.

Example of existence

Consider the stochastic differential equation in \mathcal{R}^2

$$\begin{cases} \dot{x} = -2x + y + \xi_x(t) \\ \dot{y} = -x - 2y + \xi_y(t) \end{cases} \quad (6)$$

where $\xi_x(t)$ and $\xi_y(t)$ are Gaussian noise terms, that is

$$\langle \xi_x(t) \xi_x(t') \rangle = \langle \xi_y(t) \xi_y(t') \rangle = 2\delta(t - t') .$$

The F-P equation associated to (6) is

$$\frac{\partial}{\partial t} W(\mathbf{x}, t) = -\nabla \cdot (W(\mathbf{x}, t) \mathbf{d}(\mathbf{x})) + 2\nabla^2 W(\mathbf{x}, t)$$

where the drift vector is $\mathbf{d}(\mathbf{x}) = (-2x + y, -x - 2y)$. It is easy to verify that $\mathbf{d}(\mathbf{x})$ is not a conservative field, so that detailed balance does not hold. However a stationary solution of the F-P exists, with $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ and

$$U(\mathbf{x}) = x^2 + y^2 .$$

In fact, defining $\mathbf{f} = \nabla U + \mathbf{d}$ we have

$$\mathbf{f} = (2x - (2x - y), 2y - (x + 2y)) = (y, -x)$$

and therefore equation (5) is satisfied, since

$$(\nabla - \nabla U) \cdot \mathbf{f} = \nabla \cdot \mathbf{f} - \nabla U \cdot \mathbf{f} = 2(x, y) \cdot (y, -x) = 0 .$$

Notice that in absence of noise the differential equation (6) is linear, with characteristic eigenvalues $\lambda = -2 \pm i$, and the associated trajectories are inward spirals. This makes perfectly plausible the fact that, in presence of

noise, the probability distribution of the variables is a Gaussian centered in the origin.

Remarks:

- In the linear case, that is when $d(x) = Ax$ and A is a symmetric matrix, detailed balance always holds, because $d = -\nabla U$ with $U(x) = -\frac{1}{2}xAx$. However the stationary solution $w(x) = e^{-U(x)}$ exists only if the matrix A is negative definite, that is if $w(x)$ is integrable.
- Stratonovitch (1963, p. 79) says that even when potential conditions are not met but d is a linear function of x and $K_{\alpha\beta}(x)$ are independent of x , the F-P equations can be solved. In fact it is easy to see that if $d(x) = Ax$ and A is *not* symmetric the potential conditions do not hold but the function $U(x) = -\frac{1}{2}xAx$ is a solution of equation (5).
- If the forces d_α in the Langevin equation are conservative, i.e., $d = -\nabla U$, then, if the fluctuations are thermic-like, detailed balance is satisfied and a Gibbs stationary distribution exists (Equation 4 is satisfied).
- It appears that our results may be derivable from the formulation of Graham (1980) and the more general case considered by Jauslin (1984) and Zeeman (1988). An in-depth analysis of many properties of the Fokker-Planck equation relevant for this note can be found in Tan and Wyatt (1985).

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